

Efficient computation of full Lucas sequences

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Recently, Yen and Laih (1995) proposed an algorithm to compute LUC digital signatures quickly. This signature is based on a special type of the Lucas sequence, V_k . The authors generalise their method to any type of Lucas sequence, and extend it to the 'sister' Lucas sequence, U_k . As an application, the order of an elliptic curve over $GF(2^m)$ is computed quickly.

Basic facts: In this section, we only include the minimal amount of background necessary to understand the Letter. For a systematic treatment, see [2, 3].

Let P and Q be two rational integers, and let α be a root of $x^2 - Px + Q = 0$ in the field $\mathbb{Q}(\sqrt{D})$, where $D = P^2 - 4Q$ is a non-square. Let β be the conjugate of α , i.e. $\beta = \bar{\alpha}$. The Lucas sequences $\{U_k\}_{k \geq 0}$ and $\{V_k\}_{k \geq 0}$ with parameters P and Q are given by

$$U_k(P, Q) = \frac{\alpha^k - \beta^k}{\alpha - \beta}, \quad (1)$$

$$V_k(P, Q) = \alpha^k + \beta^k. \quad (2)$$

It can easily be shown that the numbers U_i and V_i satisfy the following relations:

$$U_{i+j} = U_i V_j - Q^j U_{i-j}, \quad (3)$$

$$V_{i+j} = V_i V_j - Q^j V_{i-j}. \quad (4)$$

Fast algorithm for Lucas sequences: Assume you have to compute $U_k(P, Q)$ and $V_k(P, Q)$. If we use the binary expansion of k , it can be expressed as $k = K_0$, where

$$K_j = \sum_{i=j}^{n-1} k_i 2^{i-j}, \quad k_i \in \{0, 1\} \text{ and } k_{n-1} = 1.$$

Hence,

$$K_{j-1} = k_{j-1} + 2K_j = (K_j + k_{j-1}) + K_j. \quad (5)$$

Using Eqs. (3) and (4), we obtain

$$U_{K_{j-1}} = U_{(K_j+k_{j-1})} V_{K_j} - Q^{K_j} U_{K_{j-1}}, \quad (6)$$

$$V_{K_{j-1}} = V_{(K_j+k_{j-1})} V_{K_j} - Q^{K_j} V_{K_{j-1}}. \quad (7)$$

At iteration j , let $(l_j, h_j) = (K_j, K_j + 1)$. From Eq. (7), we see that both V_{l_j} and V_{h_j} are needed to compute $V_{l_{j-1}}$, and so to compute V_k . This is not the case for U_k , as we shall see in the following theorem.

Theorem: If k is odd, then the computation of U_k does not require the computation of U_{l_j} ($j \geq 1$).

Proof: Since k is odd (i.e. $k_0 = 1$), $U_k (= U_{l_0}) = U_{h_1} V_{l_1} - Q^{l_1}$. Thus, only the value of U_{h_1} is needed. We only need to show that the value of $U_{h_{j-1}}$ can be derived from U_{h_j} . By Eq. (5) and depending on the value of k_{j-1} , we have the following cases:

- if $k_{j-1} = 0$, then $(l_{j-1}, h_{j-1}) = (2l_j, l_j + h_j)$;
- if $k_{j-1} = 1$, then $(l_{j-1}, h_{j-1}) = (l_j + h_j, 2h_j)$.

Hence, if $k_{j-1} = 0$, then $h_{j-1} (= h_j + l_j = 2l_j + 1)$ is odd and $U_{h_{j-1}} = U_{h_j} V_{l_j} - Q^{l_j}$; otherwise, $h_{j-1} (= 2h_j)$ is even and $U_{h_{j-1}} = U_{h_j} V_{h_j}$. ■

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Inputs:  k = 2^s \sum_{i=s}^{n-1} k_i 2^{i-s}, (k_s = 1)
         P, Q
Outputs: (U_k, V_k)

U_h = 1; V_l = 2; V_h = P; Q_l = 1; Q_h = 1;
for j from n-1 to s+1 by -1
  Q_l = Q_l * Q_h;
  if k[j] == 1 then
    Q_h = Q_l * Q;
    U_h = U_h * V_h;
    V_l = V_h * V_l - P * Q_l;
    V_h = V_h * V_h - 2 * Q_h
  else
    Q_h = Q_l;
    U_h = U_h * V_l - Q_l;
    V_h = V_h * V_l - P * Q_l;
    V_l = V_l * V_l - 2 * Q_l
  fi
endfor
Q_l = Q_l * Q_h; Q_h = Q_l * Q;
U_h = U_h * V_l - Q_l;
V_l = V_h * V_l - P * Q_l;
Q_l = Q_l * Q_h;
for j from 1 to s
  U_h = U_h * V_l;
  V_l = V_l * V_l - 2 * Q_l;
  Q_l = Q_l * Q_l;
endfor
{U_k(P, Q) = U_h; V_k(P, Q) = V_l}
    
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Fig. 1 Algorithm to compute (U_k, V_k)

Remarks: (i) The presented algorithm is a left-to-right scanning one. Similar to [1], it is also possible to develop a right-to-left scanning algorithm, but that requires more temporary memories. (ii) An implementation with Pari-GP [4] is available at

`ftp://math.math.ucl.ac.be/pub/joye/pari/lucas2.gp`

Performances: The worst case of the algorithm appears when $s = 0$. Assume $s = 0$; then the computation of U_k and V_k requires $\frac{11n}{2}$ multiplications. Furthermore, only five temporary memories (with the same length as the output) are needed.

If we only want the value of V_k , the algorithm is the same as that presented on figure 1, except that we do not care of the U_h 's. Thus, the computation of V_k requires $\frac{9n}{2}$ multiplications in the worst case with only four temporary memories.

Remarks: (i) Some applications use Lucas sequences with parameter $Q = \pm 1$. In that case, the computation of U_k and V_k requires less than $3n$ multiplications with three temporary memories. (ii) If we want U_k and V_k modulo a number, the computation can be improved using the technique of the common-multiplicand [5]. (iii) Moreover, owing to the high regularity of the algorithm, it can be parallelized.

Applications: Lucas sequences have numerous applications in number theory. For example, the divisibility properties of the U_k 's (see [3, pp. 54–59]) allows to test the primality of a number N for which the factorization of $N + 1$ is partially given. It is also possible to develop efficient (pseudo) primality tests [6].

In this Letter, we shall see a less-known application. In 1985, the theory of elliptic curves emerged for cryptographic purposes. Since many cryptographic protocols [7, 8] require the knowledge of the order of an elliptic curve over $GF(2^m)$, i.e. $\#E(GF(2^m))$, we shall see how it can be computed using Lucas sequences.

Let p be a prime and $q = p^r$. Consider the elliptic curve $E/GF(q)$ such that $E(GF(q))$ has order $\#E(GF(q)) = q+1-t$. Using Weil theorem, we have

$$\#E(GF(q^l)) = q^l + 1 - \alpha^l - \beta^l, \quad (8)$$

Let $E/GF(2^r)$ be the non-supersingular elliptic curve given by the Weierstraß equation

$$E : y^2 + xy = x^3 + a_2x^2 + a_6 .$$

Assume r is small, so $t = 2^r + 1 - \#E(GF(2^r))$ can be computed by exhaustion. Moreover, if m is a multiple of l , then the curve E can be viewed as an elliptic curve over $GF(2^m)$. Hence, if we put $l = m/r$, then by Eq. (8)

$$\#E(GF(2^m)) = 2^m + 1 - V_l(t, 2^r) ,$$

where $V_l(t, 2^r)$ is the l^{th} term of the Lucas sequence $\{V_k\}$ with parameters $P = t$ and $Q = 2^r$.

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